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An Introduction to Dispersion Relations

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(Received 29 February 1964)

The concepts of causality, linearity, time symmetry, unitarity, and crossing symmetry and their relation to dispersion relations are discussed. Dispersion relations are applied to some simple electrical circuits in order to illustrate some of their properties and then a simple example is given of the use of dispersion relations for the analysis of the forward angle scattering of pions on protons.

I. INTRODUCTION

FREQUENTLY, important information about a given problem can be obtained from general physical principles without a detailed solution of the problem. For example, the laws of conservation of energy and momentum limit the possible kinematical arrangements of interacting systems; in quantum mechanics, general symmetry principles provide insight into problems otherwise difficult to assail; charge and nucleon conservation enable one to specify important features of high-energy nuclear reactions. The possibility of obtaining information about a phenomenon from general principles takes on great importance when, as is frequently true in quantum mechanics, detailed description is impossible. Such a problem is the description of scattering at high energies. Inadequate understanding, insufficient knowledge and the complexities of the calculations involved have tended to channel attention toward the general features of scattering. One such general approach is based on dispersion relations for scattering.

What dispersion relations are and how they can be used to obtain information about physical systems are illustrated in this paper. As an illustration, a dispersion relation is applied in an analysis of the scattering of charged mesons from protons. Most of what is said is of general

significance and can be applied to other problems. However, the purpose of this paper is to provide an introduction to dispersion relations and the arguments presented here need not be entirely rigorous.

This introduction is intended for those students and teachers quite unfamiliar with the concepts and use of dispersion relations. More extensive and more advanced treatments are available elsewhere in the literature. One of the best such treatments is that given by Hamilton,¹ the next article for the interested student to read after having read this one. Mandelstam² has also written a very interesting review of this subject at an advanced level. Both of these works have comprehensive bibliographies.

II. GENERAL PROPERTIES OF THE SYSTEM

Concepts which are basic to the derivation and application of dispersion relations are: causality, linearity, time symmetry, unitarity, crossing symmetry, high-energy (or high-frequency) convergence, and the properties of bound states. The physical and mathematical nature of these concepts are presented in two contexts; passing signals through electrical networks and scattering mesons by protons.

A. Causality

In Fig. 1 we show a box into which flows an input I which in turn gives rise to an output O . An essential property which we assume for every such system is causality. We assume that for any given input there is a unique output. By input and output we mean the entire entity so that

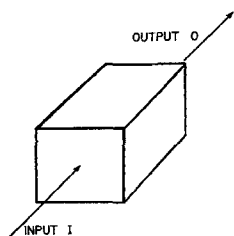


FIG. 1. A generalized system.

¹ J. Hamilton, *Progr. Nucl. Phys.* **8**, 143 (1960).

² S. Mandelstam, *Rept. Progr. Phys.* **25**, 99 (1962).

when referring to waves we mean the complete waveform extended over all time, rather than the amplitude of the waveform at some instant in time. Using this definition of input and output we also assume that if the waveforms of two different inputs are the same up to some time t , then the outputs are also the same up to that time. Note that in terms of this definition of causality a system with no input at all can have an output and still be causal. All physical systems are believed to be causal, so that it is not possible to demonstrate the content of this assumption by a comparison of systems which satisfy it with any actual one which does not. But a hypothetical noncausal system is one which possesses some forecasting device, such as a crystal ball, which could predict what the input was going to be. Then though each input, given over all time, would determine a unique output, two inputs differing only in the future behavior could then give distinct outputs.

B. Linearity

In addition to assuming causality, we limit ourselves to linear systems. A linear system is one for which any input gives rise to an output which is a linear function of the input. This statement means that if an input A gives rise to an output C and another input B gives an output D then a combined input of A+B will result in an output which is C+D. Linearity is a property of many systems, e.g., electrical circuits containing only inductors, capacitors, and resistors or mechanical systems subjected to small displacements from equilibrium.

Amplifiers for small signals are generally linear systems. If the amplifier has a gain of 2 then an input signal A comes out as 2A while an input B would come out as 2B and an input A+B would come out as 2(A+B). When an amplifier saturates, it becomes a nonlinear system for then any input pulse or combination of input pulses gives an output pulse of fixed amplitude and varying duration. To state this linearity assumption mathematically, we express the output at time t , $O(t)$, as a linear combination of the input at all previous times.

$$O(t) = \int_{-\infty}^t dt' K(t, t') I(t'). \quad (1)$$

Combining the causality assumption with the assumption of linearity, we can limit the integration in Eq. (1) to times t' prior to t , or alternatively, require that $K(t, t') = 0$ when $t < t'$.

C. Time Symmetry

The coefficients $K(t, t')$ which express $O(t)$ as a linear combination of $I(t')$ in Eq. (1) allow for a wider class of situations than we wish to consider here. We wish to restrict ourselves to those cases in which $K(t, t')$ depends only upon the time difference, $t - t'$, so that we write it as $K(t - t')$. In this case the input and output are related by

$$O(t) = \int_{-\infty}^t dt' K(t - t') I(t'). \quad (2)$$

By writing $K(t - t')$ we have assumed that only the relative time between input and output is of importance and that the instant of time at which the input went in or the output came out is not important. Consider for example a beam of light passing through a hole in the box and out the other side. Then, neglecting flight time across the box, $K(t - t') = \delta(t - t')$ and $O(t) = \int_{-\infty}^{\infty} \delta(t - t') I(t') dt'$, i.e., the light input is the light output, a relation which will be true any time of the day or year. Suppose we insert in the box a device which on Mondays and Wednesdays closes the hole through which the light passes. Then there is some explicitly time-dependent process occurring in the box and the process can not be described by $K(t - t')$; a function is needed with a more complicated dependence on t and t' . Thus by writing $K(t - t')$ we assume that there are no explicitly time-dependent processes going on in the box. This assumption results in what is called *time symmetry*, and we assume time symmetry in all the examples we consider, and in all the arguments we make.

D. Unitarity

The conditions of causality, linearity, and time symmetry restrict and help to specify certain general properties of the behavior of any system. Conservation of energy is familiar as another such condition. In the case of linear circuits conservation of energy relates the amplitudes of the incoming wave form to those of the

outgoing wave in a particularly simple fashion. The energy carried by a wave is proportional to the square of the amplitude and this is so in the case of electrical wave forms. The conservation of energy will indicate that if the input and output impedances are the same, then the sum of the squares of the amplitudes of the incoming wave must be greater than or equal to the sum of the squares of the amplitudes of the outgoing wave. A ratio of these amplitudes less than one implies that electrical energy is not conserved but that some is dissipated. This, of course, is exactly what a resistive element does. In general, we find that for any system in which energy is conserved the time average of the output energy equals or is less than the time average of the input. Thus, if

$$I(t) = A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} + A_3 e^{i\omega_2 t} + A_4 e^{-i\omega_2 t}$$

and the circuit is such that

$$O(t) = B_1 e^{i\omega_1 t} + B_2 e^{-i\omega_1 t} + B_3 e^{i\omega_2 t} + B_4 e^{-i\omega_2 t}$$

then

$$\begin{aligned} |A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 \\ \geq |B_1|^2 + |B_2|^2 + |B_3|^2 + |B_4|^2. \end{aligned}$$

If there is no dissipation or energy source in a system, then

$$\sum_{i=1} |A_i|^2 = \sum_{i=1} |B_i|^2$$

exactly. Such a relationship is a familiar one to the student acquainted with vector spaces. The A_i can be considered to be the components of a vector \mathbf{A} which is subjected to a linear transformation into a vector \mathbf{B} . That is, the action of the circuit or whatever system is in the box of Fig. 1 can be considered to be a linear transformation such that the quantity

$$\sum_i |A_i|^2$$

remains constant. The quantities

$$\sum_{i=1} |A_i|^2 \quad \text{and} \quad \sum_{i=1} |B_i|^2$$

represent the lengths of these vectors. Thus when energy is conserved and there is no dissipation we are interested only in those linear transformations which do not change the lengths of vectors. Such transformations are said to be unitary

transformations. In quantum mechanics the square magnitudes of the amplitudes with which we deal are probabilities, and probability is also a conserved quantity. Because of this fact we restrict ourselves by the additional condition of unitarity. As a consequence of unitarity we are always able to assume

$$\sum_{i=1} |A_i|^2 = \sum_{i=1} |B_i|^2,$$

whether the A_i refer to the voltages of the component sine waves of the input to an electrical circuit or the probability amplitudes of some particular quantum-mechanical state of a system.

III. PROPERTIES OF THE KERNEL

A. Physical Significance

We have mentioned that the quantity $K(t-t')$ appearing in Eq. (1) represents the effect of the system on the phase and amplitude of the input. The physical significance of $K(t-t')$ is most clearly seen if we consider a delta function input as illustrated in Fig. 2(a). Integration gives $K(t) = O(t)$, the output corresponding to the delta function input, and Fig. 2 illustrates a possible output. Note that $K(t)$ is zero up until time t' as demanded by causality. For a mechanical system the input and response illustrated in Fig. 2 can be interpreted as a sharp blow causing the system to undergo rapidly damped vibrations.

B. Frequency Response of the System

In any linear system there is a simple relation between the time response and the frequency response of the system. Before seeing that this is so, let us simplify Eq. (2) by defining $K(t-t') = 0$ for $t' > t$; then Eq. (2) can be written as

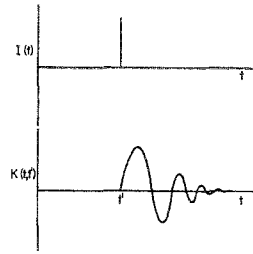
$$O(t) = \int_{-\infty}^{\infty} dt' K(t-t') I(t'). \quad (3)$$

If we consider a periodic input $e^{-i\omega t}$ then Eq. (3) becomes

$$O(t) = \int_{-\infty}^{\infty} dt' K(t-t') e^{-i\omega t'}. \quad (4)$$

This expression is of the form $k(\omega) e^{-i\omega t}$, where

FIG. 2. A pulse input resulting in an output which oscillates and decays with time.



$k(\omega)$ is the Fourier transform of the input so that we have

$$k(\omega) = \int_{-\infty}^{\infty} dt' K(t-t') e^{+i\omega(t-t')}, \quad (5)$$

relating the behavior of the system in time to the frequency response of the system. Anyone who has studied electric-circuit theory knows how important it is to know the response of a circuit as a function of the frequency of the input voltage. In the quantum mechanics of scattering similar relations are also of a great importance. However, in quantum mechanics the input and output amplitudes correspond to probability amplitudes and the frequency corresponds to the energy of the input and output.

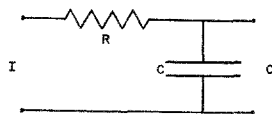
IV. ILLUSTRATIONS AND COMPLEX VARIABLES

A. Causality and an Electric Circuit

General restrictions have been placed on $K(t-t')$ by the assumptions of causality, linearity, etc. For example, causality demands that $K(t-t')$ be zero for $t' > t$. This limitation results in a corresponding limitation on the frequency response, because $k(\omega)$ is related to $K(t-t')$ by Eq. (5). The limitation on frequency response, i.e., the condition which $k(\omega)$ satisfies corresponding to $K(t-t') = 0$ for $t' > t$, determines the dispersion relations characterizing the system.

To see what properties of $k(\omega)$ correspond to the causality condition, consider some simple electrical circuits. In Fig. 3 is shown an integrating circuit. From linear circuit theory we know

FIG. 3. An integrating circuit.



that the frequency response of this circuit is expressed by

$$k(\omega) = (i/\omega C) / [R + (i/\omega C)], \quad (6)$$

where $k(\omega)$ is the fraction of input appearing at the output. Does this expression satisfy the causality relation that arises from the demand that $K(t-t') = 0$ for $t' > t$?

In order to investigate whether it does or not let us express $K(t)$ in terms of $k(\omega)$ (where for convenience $t-t'$ has been replaced by t). This is easily done by the inverse Fourier transform of Eq. (3).

$$K(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega k(\omega) e^{-i\omega t}. \quad (7)$$

It is also convenient to rewrite Eq. (6) as

$$k(\omega) = (i/RC) [\omega + (i/RC)]^{-1}. \quad (8)$$

Then the time response of the circuit is

$$K(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega i/RC \frac{1}{\omega + (i/RC)} e^{-i\omega t}. \quad (9)$$

It is the evaluation of this type of integral which illustrates a result very important to the determination of dispersion relations. Such an integral can be readily evaluated by contour integration in the complex plane, and since an understanding of contour integration is essential to an understanding of dispersion relations, the subject is discussed here in some detail.³

Frequent use is made of the terms residue, pole, and singularity so let us first define these terms. Consider a function $f(\omega)/p(\omega)$ where $f(\omega)$ is analytic (e.g., a polynomial) and $p(\omega)$ is a polynomial of order n . The polynomial $p(\omega)$ can be factored into its roots so that the function could be written $f(\omega)/[(\omega-a_1)\cdots(\omega-a_n)]$, where a_1, \cdots, a_n are the n roots of $p(\omega)$. Whenever $\omega = a_1$ or $a_2 \cdots$ or a_n the function $[f(\omega)/p(\omega)]$ becomes undefined. When $f(\omega)$ is defined and nonzero for $\omega = a_1, a_2, \cdots$, the function $f(\omega)/p(\omega)$ is then said to have a singularity, or pole, at $\omega = a_1, a_2 \cdots a_n$. If the important assumption is made that no two roots of $p(\omega)$ are alike, it is easy and useful to define a quantity called the

³ R. V. Churchill, *Introduction to Complex Variables and Applications* (McGraw-Hill Book Company, Inc., New York, 1948), pp. 116 *et seq.*

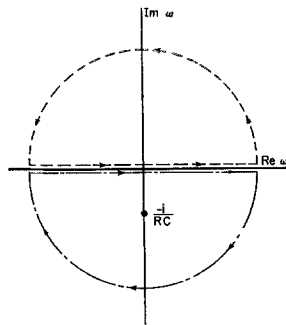


FIG. 4. Possible contours of integration.

residue of the function at a pole. For example, consider the pole at $\omega = a_j$; the residue is obtained by multiplying the function $f(\omega)/p(\omega)$ by $(\omega - a_j)$ and then substituting $\omega = a_j$ so the residue evaluated at the pole a_j is

$$\frac{f(a_j)}{(a_j - a_1) \cdots (a_j - a_{j-1})(a_j - a_{j+1}) \cdots (a_j - a_n)},$$

where the factor which produced the singularity has been cancelled by our multiplication by $(\omega - a_j)$. A similar procedure gives the residue of the function at any other pole. It should be noted that the roots a_i need not be real, and consequently they can be points anywhere in the complex plane.

The preceding ideas are very useful in evaluation of integrals of the type concerning us. If as $\omega \rightarrow \infty$ the numerator of our integrand goes to zero at least as fast as $1/\omega$, then it can be proven by the theory of complex integration that an integral of the form (where a is real)

$$\int_{-\infty}^{\infty} d\omega \frac{f(\omega)e^{-i\omega t}}{(\omega - ia)} \quad (10)$$

can be evaluated by integrating along a path or contour which passes along the real axis and then in a semicircle through the half of the complex plane in which it goes to zero. Such a contour is indicated in Fig. (4) by a dashed line. The value of such an integral is given by the product of $\pm 2\pi i$ (– if the contour is clockwise, + if it is counterclockwise) with the sum of all the residues of $f(\omega)e^{-i\omega t}/(\omega - ia)$ at the poles enclosed by the contour. The integral would be $2\pi i f(ia)e^{+at}$ if the pole at $\omega = ia$ is the only pole in the upper half-plane. Applying these ideas to Eq. (7) it is apparent that the integrand has one pole at

$\omega = -1/RC$, i.e., the pole lies on the negative imaginary axis of Fig. 4 as indicated by the dot.

The key point in what has been said is that the half-plane chosen for the contour must be the one in which $|e^{-i\omega t}|$ is bounded. If $t < 0$ then we must choose the half-plane in which the imaginary part of ω is positive. Writing $\omega = x + iy$ we have $|e^{-i\omega t}| = |e^{-i(x+iy)t}| = |e^{+yt}|$ which indeed goes to zero for large positive y and negative t . Consequently for $t < 0$ the integral must always be taken along a contour in the upper half-plane where y is positive.

Using contour integration we evaluate the integral of Eq. (7) by the method of residues, but there are no poles in the upper half-plane so the sum of the residues is zero.

Consequently $K(t) = 0$ for $t < 0$. This is the result we expected; it is the condition imposed on the system by causality. However, we now see what is the corresponding condition on $k(\omega)$; $k(\omega)$ must not have any poles in the upper half-plane if causality is to be satisfied in general. We shall see that this restriction on $k(\omega)$ is essential to specifying dispersion relations.

To tie up the loose ends of the example consider the case of $t > 0$. Then the contour chosen must pass through the lower half-plane where it encloses the pole at $\omega = -i/RC$; thus there is a nonzero residue. By multiplying the residue by $-2\pi i$ (minus because the path of integration runs around clockwise) $K(t)$ is found to be

$$K(t) = (1/RC)e^{-t/RC}, t > 0, \quad (11)$$

which is the expected output of this circuit for a delta function input.

B. High-Frequency Behavior

Although the essential point that $k(\omega)$ may have no poles in the upper half-plane has been adduced, further examples may be considered profitably. For instance, let us interchange the capacitor and the resistor to obtain the circuit shown in Fig. 5.

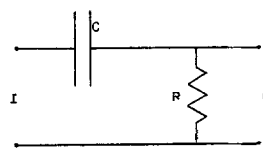
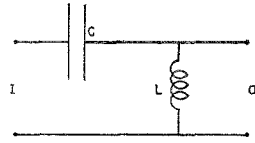


FIG. 5. A differentiating circuit.

FIG. 6. An LC circuit.



The frequency response can be written as

$$k(\omega) = R/[R + (i/\omega C)] = \omega[\omega + (i/RC)]^{-1}. \quad (12)$$

There is still a pole at $\omega = -i/RC$ but the integrand has changed form. By contour integration $K(t)$ is found to be

$$K(t) = -(1/RC)e^{-t/RC} \quad \text{for } t > 0.$$

How in such a simple circuit does a positive delta function input result in a negative output? A quick examination of the behavior of $k(\omega)$ in Eq. (12) at high frequencies reveals the difficulty. The integrand does not go to zero as ω becomes large, i.e., there are convergence problems. In effect our analysis has been incomplete for by neglecting the high-frequency behavior we have considered only $t < 0$ and $t > 0$ while neglecting $t = 0$. That this statement is so can be seen by rewriting Eq. (10) as

$$k(\omega) = \omega/[\omega + (i/RC)] \\ = 1 - (i/RC)[\omega + (i/RC)]^{-1}. \quad (13)$$

Then $K(t)$ becomes

$$K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left[1 - \frac{i}{RC} \frac{1}{\omega + i/RC} \right] e^{i\omega t} \\ = \delta(t) - (1/RC)e^{-t/RC}. \quad (14)$$

The first term of the integrand gives the delta function while the second term gives the negative of Eq. (11).

This particular example serves to illustrate two features of interest. Even though two functions may have a pole at the same point, differing convergence properties, i.e., different behavior at high frequencies can result in different time behavior. One of the recurring problems of dispersion relations is the analysis of the convergence properties of functions at high frequencies, i.e., at $t = 0$. Generally, the integrand can be treated as in Eq. (13) and broken into two parts, a $t = 0$ part, and a part corresponding

to $t > 0$, which in our special case give rise to a positive pulse with a negative overshoot.

C. Resonances

A third circuit, shown in Fig. 6, exhibits other features of $k(\omega)$ which are related to scattering. Proceeding as before we find $k(\omega)$

$$k(\omega) = (\omega L/i)/[(\omega L/i) + (i/\omega C)] \\ = \omega^2/(\omega^2 - 1/LC), \quad (15)$$

and then rewrite Eq. (10) in a form convenient for Fourier transformation

$$k(\omega) = 1 + \frac{1}{2(LC)^{1/2}} \left[\frac{1}{\omega - (LC)^{-1/2}} - \frac{1}{\omega + (LC)^{-1/2}} \right]. \quad (16)$$

$K(t)$ is then obtained by Fourier transformation of $k(\omega)$ and contour integration:

$$K(t) = \delta(t) - \frac{i}{2(LC)^{1/2}} \left[e^{it/(LC)^{1/2}} - e^{-it/(LC)^{1/2}} \right]. \quad (17)$$

It is no surprise that this expression is simply the undamped oscillation characteristic of an excited resonant circuit.

Expressed in terms of the complex plane as shown in Fig. 7, the closer the poles are to the real axis the more resonant is the behavior of the circuit. When, as in the present example, the poles lie on the real axis, there is undamped resonance.

For the case of particle scattering, the resonances correspond to resonances in the cross section. If the poles lie on the real axis, then the resulting undamped resonance represents the formation of a stable particle, the lifetime of the resonance decreasing as the pole is displaced from the real axis.

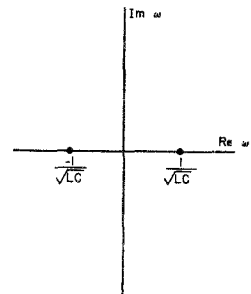


FIG. 7. The poles of an LC circuit.

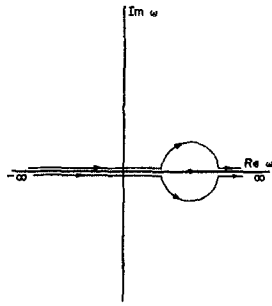


FIG. 8. Paths of integration in calculating the Cauchy principal value of an integral.

D. Negative Frequencies and Crossing Symmetry

When dealing with a specific problem, it is frequently useful to observe the symmetry properties of the system under consideration. In the case of electric circuits the time response of a circuit, characterized by $K(t)$, must be a real quantity, i.e., $K(t) = K^*(t)$. From this fact we can use Eq. (7) to write

$$\begin{aligned} \int_{-\infty}^{\infty} k(\omega) e^{-i\omega t} d\omega &= \int_{-\infty}^{\infty} k^*(\omega) e^{+i\omega t} d\omega \\ &= \int_{-\infty}^{\infty} k^*(-\omega) e^{-i\omega t} d\omega \end{aligned}$$

and it follows that $k(\omega) = k^*(-\omega)$. Consequently there is a symmetry of reflection across the imaginary axis. This symmetry is well illustrated in Eqs. (6), (12), and (15) and the corresponding figures.

In the case of quantum-mechanical scattering $K(t)$ need not be real and this particular symmetry property can not be used. Nevertheless, we find an analogous symmetry called crossing symmetry which is essential for the application of dispersion relations to scattering theory.

V. DISPERSION RELATIONS

A. Cauchy Principal Value

In the LC circuit example the contour of integration passed directly through, rather than around the two poles on the real axis and our simple rules for integration did not apply. Nevertheless, applying these rules to the LC circuit we obtain the correct value of the integral. A more correct way to get the answer would be to introduce some resistance into the circuit and

thus shift the poles off the real axis. Then our rules of contour integration could be applied to a contour passing along the real axis to obtain an answer which depended on the resistance. Taking the limit as the resistance became negligibly small we would obtain Eq. (17). This result is not general but is a consequence of the fortuitous cancellation of the contributions from the two poles. If there were only one pole it is clear that the problem of integrating through a singularity could not be so easily dealt with. In fact in such a case the value of the integral as we normally think of it does not exist.

It turns out to be very useful to specify a different type of contour integration which assigns to such integrals a unique value. As a special case suppose we have an integrand of the form $k(\omega)/(\omega-a)$, where $k(\omega)$ goes to 0 fast enough to avoid convergence problems when $|\omega| \rightarrow \infty$ and has no poles in the upper half-plane and where a is real. If we treat the problem of integrating

$$\int_{-\infty}^{\infty} d\omega \frac{k(\omega)}{\omega-a}$$

as one of complex integration from $-\infty$ to $+\infty$, we can dodge around the pole at $\omega=a$. A possible course of action is indicated in Fig. 8 where the path of integration is taken along the real axis and a detour is made into the upper half-plane; a second path is indicated in which the detour is made into the lower half-plane.

If we take the sum of these two integrals and divide by 2 (to compensate for the doubling of the integral by the addition) we obtain the Cauchy principal value of the integral, usually written

$$\mathcal{P} \int_{-\infty}^{\infty} d\omega \frac{k(\omega)}{(\omega-a)}.$$

The assumptions of convergence of $k(\omega)$ and the absence of poles in the upper half-plane assure the existence of the principal value and satisfy the conditions for validity of the easily proven theorem which says the Cauchy principal value is given by πi times the residue of the integrand:

$$\mathcal{P} \int_{-\infty}^{\infty} d\omega \frac{k(\omega)}{\omega-a} = \pi i k(a). \quad (18)$$

B. Dispersion Relations

What is the utility of the Cauchy principal value? It provides a unique relation between the real and imaginary parts of the function under consideration. From Eq. (18) it follows at once that

$$\operatorname{Re} k(\omega) = (1/\pi) \mathcal{P} \int_{-\infty}^{\infty} d\omega' [\operatorname{Im} k(\omega') / (\omega' - \omega)] \quad (19a)$$

$$\operatorname{Im} k(\omega) = -(1/\pi) \mathcal{P} \int_{-\infty}^{\infty} d\omega' [\operatorname{Re} k(\omega') / (\omega' - \omega)]. \quad (19b)$$

Such relations between the real and imaginary parts of a function are called dispersion relations.

For Eq. (19) to be valid it is necessary that there be no poles in the upper half-plane. This condition is guaranteed for electric circuits by the assumption of causality. It is also necessary that $k(\omega) \rightarrow 0$ as $1/|\omega|$ for large values of $|\omega|$. Later we consider what to do when the function under consideration does not go to zero fast enough.

Since all the necessary conditions are satisfied by linear electric circuits we have demonstrated the existence of dispersion relations Eq. (19) for such circuits. Let us now see how such relations may be used to extract information from experimental data.

C. Power Input

In a linear circuit the current and applied voltage are related by an expression of the form

$$Ie^{-i\omega t} = Ee^{-i\omega t} Y(\omega),$$

where I and E are the maximum amplitudes of the input current and voltage and $Y(\omega)$ is the complex admittance of the circuit. For such a circuit the time average of the power input is

$$P = \frac{1}{2} E^2 \operatorname{Re} Y(\omega).$$

Thus the $\operatorname{Re} Y(\omega)$ can be determined by measuring the power input into such a circuit. If it is measured as a function of frequency, the dispersion relation Eq. (19b) can be used to find $\operatorname{Im} Y(\omega)$.

Thus by one relatively simple experiment it is possible to determine the complete form of the admittance $Y(\omega)$.

VI. PION-NUCLEON SCATTERING

A. Linearity

The detailed derivation of scattering theory in terms of input and output functions linearly related by a kernel (Green's function) can be found in the many textbooks on quantum mechanics.⁴ Consequently, what follows serves only as a review of the results of this theory. For an incident plane wave e^{ikx} (where k is the propagation vector) the total wavefunction will be asymptotically

$$\psi = e^{ikx} + \frac{f(\omega, \theta, \varphi)}{r} e^{ikr}, \quad (20)$$

where r , θ , and φ are spherical coordinates with origin at the scattering center. The quantity ω is the frequency of the incident beam and is related to the energy of the incident particles by $E = \hbar\omega$. For scattering angles other than zero, only the magnitude of $f(\omega, \theta, \varphi)$ is directly observable, but as we see, $f(\omega, 0, 0)$ has a more direct physical significance. For purposes of simplicity, we restrict our considerations to forward angle scattering $\theta = 0^\circ$, and let $f(\omega)$ denote the value of $f(\omega, \theta, \varphi)$ when θ is zero. Then, $f(\omega)$ is analogous to the $k(\omega)$ in our treatment of electric circuits and because of causality it too has no poles in the upper half-plane.

The forward-scattering amplitude $f(\omega)$ is itself closely related to the experimentally observable quantity $\sigma(\omega)$, the 0° differential cross section. In fact

$$|f(\omega)|^2 = \sigma(\omega) \big|_{\theta=0}. \quad (21)$$

Consequently, the application of a dispersion relation to $f(\omega)$ may yield useful information about $\sigma(\omega)$.

B. Unitarity: The Optical Theorem

By restricting ourselves to 0° we can make use of another very useful property of $f(\omega)$ that follows from unitarity. If we examine Eq. (20), we see that the plane wave term e^{ikx} not only accounts for the incoming plane wave, but continues on past the scattering center with undiminished amplitude. Since in fact, the forward

⁴ For example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), pp. 162-164.

beam is attenuated not only by scattering, but possibly also by absorption, this must be accounted for by interference between the two terms. The attenuation is determined by the total cross section $\sigma_T(\omega)$, while the interference is determined by $f(\omega)$, and hence we expect a relationship between these. Since this relation was derived for light waves prior to the development of quantum mechanics, it is called the optical theorem. It can be deduced in various ways. One is by a straightforward calculation of the interference effects between the plane wave, e^{ikx} and the outgoing wave, $f(\omega, \theta, \varphi)e^{ikr}/r$ such as is done by Van de Hulst.⁵ Another method which exhibits this theorem more directly as a consequence of unitarity is to decompose the plane wave, e^{ikx} into a sum of incoming and outgoing spherical waves. Then, the magnitude of the incoming wave part of e^{ikx} must equal that of the combined outgoing waves from the plane wave and the outgoing spherical wave, plus the probability for nonelastic processes.⁶ The result of either of these processes is that

$$\text{Im} f(\omega) = (k/4\pi)\sigma_T(\omega), \quad (22)$$

where k is the wave number of the incident particles.

C. Crossing Symmetry

In the case of the electrical circuits we know that $K(t)$ must be real. As a consequence we found that $k(\omega) = k^*(-\omega)$. In the scattering of neutral particles it turns out that the relation $f(\omega) = f^*(-\omega)$ holds; but for charged particles it is not valid. Nevertheless, there is a very useful symmetry known as crossing symmetry which applies to the case of charged particles. The particular type of scattering under consideration is that of charged pions on protons. Since there are two kinds of charged pions π^+ and π^- there will be a different scattering amplitude associated with each. Denoting these as $f_+(\omega)$ and $f_-(\omega)$, respectively, the symmetry relation of interest is

$$f_+(\omega) = f_-^*(-\omega), \quad f_-(\omega) = f_+^*(-\omega). \quad (23)$$

This relation can be based on the operator

⁵ H. C. Van de Hulst, *Scattering of Light by Small Particles* (John Wiley & Sons, Inc., New York, 1957), p. 30.
⁶ Reference 4, pp. 105–106.

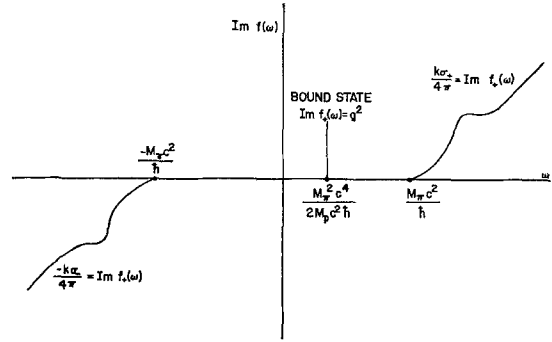


FIG. 9. Schematic representation of the imaginary part of the π^+-p scattering amplitude.

correspondences

$$p_x \rightarrow -i\hbar \frac{\partial}{\partial x} - e \frac{A_x}{c} \quad \text{and} \quad E \rightarrow i\hbar \frac{\partial}{\partial t} - e\varphi,$$

(A_x is the x component of the electromagnetic vector potential and φ is the scalar potential) in terms of which Schrödinger equation can be written

$$\left(-i\hbar \frac{\partial}{\partial x} - e \frac{A_x}{c}\right)^2 \psi = \left(i\hbar \frac{\partial}{\partial t} - e\varphi\right) \psi. \quad (24)$$

Complex conjugation together with a change of the sign of the charge will give

$$\left(i\hbar \frac{\partial}{\partial x} + e \frac{A_x}{c}\right)^2 \psi^* = -\left(i\hbar \frac{\partial}{\partial t} - e\varphi\right) \psi^*. \quad (25)$$

This result implies that

$$f_+(\omega) = f_-^*(-\omega); \quad f_-(\omega) = f_+^*(-\omega) \quad (23)$$

since the effect of the above procedure is to leave the left-hand operator of Eq. (24) unchanged and to change the sign of the energy operator and produce the complex conjugate of the wavefunction. The transformation from Eq. (24) to Eq. (25) is known as charge conjugation.⁷ The result expressed in Eq. (23) is often referred to as the crossing theorem.

Crossing symmetry is useful because in the case of the scattering of π^+ on protons it is of interest to examine the properties of the scattering amplitude $f_+(\omega)$ in the complex ω -plane. However, $f_+(\omega)$ is defined only for $\hbar\omega \geq M_\pi c^2$ (see

⁷ S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, White Plains, New York, 1955), Vol. 1, pp. 141–144.

Fig. 9) because the energy of the incident wave must be great enough for the pion to exist (M_π is the pion rest mass). Fortunately the crossing theorem Eq. (23) makes possible the definition of $f_+(\omega)$ for negative frequencies in terms of the scattering amplitude representing π^- scattering. Thus $f_+(\omega)$ can be defined for unphysical values of $\omega \leq (-M_\pi c^2/\hbar)$ as well as for values of $\omega \geq (+M_\pi c^2/\hbar)$ (see Fig. 9) in terms of the quantity $f_-^*(-\omega)$ which is directly related to physically real phenomena.

D. Pion-Nucleon Bound State

There is still the intermediate region

$$(-M_\pi c^2/\hbar) < \omega < (M_\pi c^2/\hbar),$$

where $f_+(\omega)$ is undefined. This region corresponds to energies of pions which have formed bound states with the proton. Such a bound state does exist; it is commonly called the neutron ($\pi^- + p = n$). By a heuristic argument the pion energy $\hbar\omega$ corresponding to this bound state can be found. Consider a pion with energy $\hbar\omega$ and a stationary proton with energy $M_p c^2$, where M_p is the proton rest mass. The momentum of this system will satisfy

$$\hbar^2 \omega^2 - M_\pi^2 c^4 = p^2 c^2 \quad (26)$$

and its total energy will be

$$E = \hbar\omega + M_p c^2. \quad (27)$$

From (26) and (27) the total energy in the c.m. system $E_{c.m.}$ can be calculated as

$$\begin{aligned} E_{c.m.}^2 &= E^2 - p^2 c^2 \\ &= (\hbar\omega + M_p c^2)^2 - (\hbar^2 \omega^2 - M_\pi^2 c^4). \end{aligned} \quad (28)$$

Then if there is a resonance corresponding to the neutron (rest mass M_n) the resonance energy will correspond to the neutron rest energy

$$\begin{aligned} E_{c.m.}^2 &= M_n^2 c^4 = (\hbar\omega + M_p c^2)^2 \\ &\quad - (\hbar^2 \omega^2 - M_\pi^2 c^4). \end{aligned} \quad (29)$$

Making the approximation $M_n = M_p$, solution of Eq. (29) for $\hbar\omega$ in terms of M_p and M_π gives

$$\hbar\omega = -M_\pi^2 c^4 / 2M_p c^2. \quad (30)$$

Equation (30) specifies the value of ω for which $f_-(\omega)$ has a bound state, but we are interested in $f_+(\omega)$. Since $f_+(\omega) = f_-^*(-\omega)$ it is

clear that the relevant frequency is

$$\hbar\omega = (M_\pi^2 c^4 / 2M_p c^2)$$

as indicated in Fig. 9.

So far all that has been done is to define $f_+(\omega)$ for all values of ω . To be useful $f_+(\omega)$ must be related to experimental data and it is in this respect that the utility of the optical theorem [Eq. (22)] becomes apparent. From measurements of the total cross section σ_{T^+} of π^+ on protons the imaginary part of $f_+(\omega)$ can be obtained immediately as $k\sigma_{T^+}/4\pi$ for the region of $\omega \geq M_\pi c^2/\hbar$ while Eq. (23) implies that measurements of the total cross section σ_{T^-} of π^- scattered on protons give $\text{Im } f_+(\omega) = -k\sigma_{T^-}/4\pi$ for $\omega \leq -M_\pi c^2/\hbar$ as shown schematically in Fig. 9. Consequently by measurements of the total cross sections of the scattering of π^+ and π^- on protons the imaginary part of the scattering amplitude can be experimentally determined.

The value of $\text{Im } f_+(\omega)$ corresponding to the bound state is determined from considerations of field theory to be a quantity called the pion-nucleon coupling constant g^2 . This quantity is not accurately known and knowledge of its value is an important problem in field theory. Knowledge of g^2 and the two total cross sections makes it possible to specify $\text{Im } f_+(\omega)$ for all values of the incident energy $\hbar\omega$, or conversely, g^2 can be determined from the scattering cross sections.

E. Dispersion Relations

The essential features of what has so far been developed are that:

(1) the general principle of causality implies that the scattering amplitude has no poles in the upper half of the complex plane.

(2) Crossing symmetry makes it possible to define the $\pi^+ - p$ scattering amplitude $f_+(\omega)$ for all values of ω .

(3) The optical theorem provides a means of determining experimentally the imaginary part of $f_+(\omega)$ by measuring the total cross sections σ_{T^+} and σ_{T^-} for charged pions scattered from protons.

(4) The imaginary part of the scattering amplitude at the frequency corresponding to the bound state is the pion-nucleon coupling constant g^2 .

Aside from questions of convergence these conditions are sufficient to assure the existence of dispersion relations. In other words $f_+(\omega)$ can be completely determined from a dispersion relation and measurements of the total cross sections. Since $f_+(\omega)$ completely specifies the differential cross section in the forward direction, $\sigma(\omega)$, the values of $f_+(\omega)$ determined from theory and the measured total cross sections can be compared with values of $f_+(\omega)$ implied by the measured differential cross section $\sigma(\omega)$. From Eq. (19) it follows that

$$\mathcal{P} \int_{-\infty}^{\infty} d\omega' [\text{Im } f_+(\omega') / (\omega' - \omega)] = \pi \text{Re } f_+(\omega). \quad (31)$$

Performing this integration we can obtain the $\text{Re } f_+(\omega)$ for any value of ω . In this manner the real part of the scattering amplitude can be obtained from the imaginary part which in turn can be obtained from measurements of the total cross sections for π^+ and π^- scattering on protons.

The role of causality is to guarantee the absence of poles in the upper half-plane and thus to assure us that an essential condition for the validity of Eq. (31) is in fact satisfied.

F. Convergence

There is one important additional point which the above arguments have avoided. What if the function $\text{Im } f_+(\omega)$ does not go to zero fast enough as $|\omega| \rightarrow \infty$ to assure the existence of the Cauchy principal value of the integral (as it did not for one of the electrical circuits which was considered)? In order for the principal value to exist the integrand must go to zero at least as $1/|\omega|^2$ for large $|\omega|$ or equivalently $\text{Im } f_+(\omega)$ must go to zero as $1/|\omega|$ for large $|\omega|$. In fact all evidence indicates $\text{Im } f_+(\omega)$ does not go to zero at all let alone as $1/\omega$. Actually this is a simple problem to solve. We simply put additional factors in the denominator of the integrand until the integrand goes to zero as $1/|\omega|^2$. The Cauchy principal value then has the form

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im } f_+(\omega)}{(\omega-a)(\omega-b)(\omega-c)} = \pi \left[\frac{\text{Re } f_+(a)}{(a-b)(a-c)} + \frac{\text{Re } f_+(b)}{(b-a)(b-c)} + \frac{\text{Re } f_+(c)}{(c-a)(c-b)} \right]. \quad (32)$$

The use of only three factors in the denominators of the example is based on another important assumption we now make. It is assumed that σ_{π^+} and σ_{π^-} are bounded. This means that for sufficiently large energies $\text{Im } f_+(\omega) = k\sigma_{\pi^+}/4\pi$ increases approximately linearly with the wave number k . At sufficiently large energies k is proportional to ω , consequently at large energies $\text{Im } f_+(\omega)$ grows as ω and proper convergence of the integrand can be assured by dividing $\text{Im } f_+(\omega)$ by three factors of ω .

The choice of the values of ω at which the residues are to be evaluated is dictated by convenience. Two of the points usually chosen are

$$\omega = \pm \mu = \pm M_{\pi}c^2/\hbar. \quad (33)$$

The quantity μ is introduced in Eq. (33) for notational convenience. It represents the frequency corresponding to the pion-rest mass, and the quantities $\text{Re } f_+(\mu)$ or $\text{Re } f_-(\mu)$ are known as the scattering lengths. A third point ω is needed to evaluate the integral and this is the one which we vary. Constructing and evaluating the integral as in Eq. (32) we have

$$\mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im } f_+(\omega')}{(\omega' + \mu)(\omega' - \mu)(\omega' - \omega)} = \pi \text{Re} \left[\frac{f_+(-\mu)}{2\mu(\mu + \omega)} + \frac{f_+(\mu)}{2\mu(\mu - \omega)} + \frac{f_+(\omega)}{(\omega^2 - \mu^2)} \right]. \quad (34)$$

Since $\text{Re } f_+(-\mu)$ is not directly related to an observable quantity let us make use of the crossing symmetry to replace it with the real part of the scattering amplitude of the charge conjugate particle. Then rearranging terms we obtain

$$\text{Re } f_+(\omega) = \frac{(\omega^2 - \mu^2)}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im } f_+(\omega')}{(\omega'^2 - \mu^2)(\omega' - \omega)} - \frac{\text{Re } f_-(\mu)(\omega - \mu)}{2\mu} + \frac{\text{Re } f_+(\mu)(\omega + \mu)}{2\mu}. \quad (35)$$

This result expresses $\text{Re } f_+(\omega)$ in terms of the scattering lengths and an integral which can be evaluated in terms of available experimental data. To see how this evaluation can be performed, let us replace $\text{Im } f_+(\omega')$ in Eq. (35) by the quantity $(k/4\pi)\sigma_T^+(\omega')$. In the range $-\infty < \omega' \leq -\mu$, $\sigma_T^+(\omega') = \sigma_T^-(-\omega)$, the total cross section for π^- scattering on protons at positive energies, while in the range $\mu \leq \omega' < \infty$ $\sigma_T^+(\omega')$ is directly measurable as the total $\pi^+ - p$ cross section. The bound state is taken into account by including the delta-function term $2\pi g^2 \delta[\omega' - (\mu^2 \hbar / 2M_p c^2)]$ in the definition of $\text{Im } f_+(\omega')$. With the above substitutions the integral in Eq. (35) becomes

$$\begin{aligned} \frac{(\omega^2 - \mu^2)}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im } f_+(\omega')}{(\omega'^2 - \mu^2)(\omega' - \omega)} &= \frac{\omega^2 - \mu^2}{4\pi^2} \int_{-\infty}^{-\mu} d\omega' \frac{k\sigma_T^-(-\omega)}{(\omega'^2 - \mu^2)(\omega' - \omega)} \\ &+ \frac{\omega^2 - \mu^2}{4\pi^2} \int_{\mu}^{\infty} d\omega' \frac{k\sigma_T^+(\omega')}{(\omega'^2 - \mu^2)(\omega' - \omega)} + \frac{2g^2(\omega^2 - \mu^2)}{\mu^2[\omega - (\mu^2 \hbar / 2M_p c^2)][1 - (\mu \hbar / 2M_p c^2)^2]} \\ &= \frac{\omega^2 - \mu^2}{4\pi^2} \int_{\mu}^{\infty} d\omega' \frac{k}{(\omega'^2 - \mu^2)} \left[\frac{\sigma_T^+(\omega')}{(\omega' - \omega)} + \frac{\sigma_T^-(\omega')}{\omega' + \omega} \right] + \frac{2g^2(\omega^2 - \mu^2)}{\mu^2[\omega - (\mu^2 \hbar / 2M_p c^2)][1 - (\mu \hbar / 2M_p c^2)^2]}, \quad (36) \end{aligned}$$

where in the last step the transformation $\omega' \rightarrow -\omega'$ was made in the integral with negative limits in order to obtain a more convenient form. Equation (36) can be evaluated numerically from measured total cross sections, thereby providing sufficient information (in conjunction with the coupling constant and scattering lengths) for the evaluation of $\text{Re } f_+(\omega)$, and consequently of the 0° differential cross section

$$\sigma_+(\omega) = |f_+(\omega)|^2.$$

Existing experimental values for the scattering lengths are uncertain. Determination of these parameters necessitates measurement of cross sections at low energies and, at low energies, Coulomb effects make interpretation of the data difficult. Consequently, rather than use experimental values, the scattering lengths are often left as free parameters. Similarly the value of g^2 is not well known and it too is left as a free parameter. Thus, one procedure for analyzing scattering data is to use the measured total cross sections [as in Eq. (29)] and to adjust the three parameters $\text{Re } f_+(\mu)$, $\text{Re } f_-(\mu)$, and g^2 to obtain the best fit to the 0° differential cross section. The quality of the fit is then an indication of the validity of the underlying fundamental assumptions of causality, unitarity, crossing symmetry, and boundedness of the total cross section at high energies. In addition, the values of the parameters used to fit the data can be compared with their experimental values to see if they are in reasonable agreement.

A number of authors have made such fits. Generally the scattering lengths are obtained from a phase-shift analysis of low-energy pion scattering data and are expressed in terms of the quantities a_1 and a_3 where

$$\text{Re } f_+(\mu) = \lambda_c(1 + \hbar\mu/M_p c^2)a_3$$

and

$$\text{Re } f_-(\mu) = \lambda_c(1 + \hbar\mu/M_p c^2)(2/3a_1 + 1/3a_3). \quad (37)$$

(λ_c is the pion Compton wavelength $\hbar/M_\pi c$). Hamilton and Woolcock⁸ have obtained the values $a_1 = 0.178$ and $a_3 = 0.087$. Although better values are now available⁹ we have illustrated this example with theoretical curves which Spearman¹⁰ has constructed from these earlier values, and a coupling constant $g^2 = 0.08$ and total

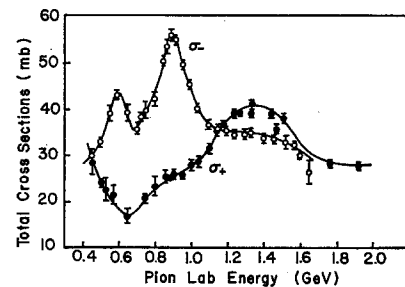


FIG. 10. The total cross sections for the scattering of π^- from protons (open circles) and π^+ from protons (solid circles) taken from Hamilton.⁹

⁸ J. Hamilton and W. S. Woolcock, Phys. Rev. **118**, 291 (1960).

⁹ J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

¹⁰ T. D. Spearman, Nuovo Cimento **15**, 147 (1960).

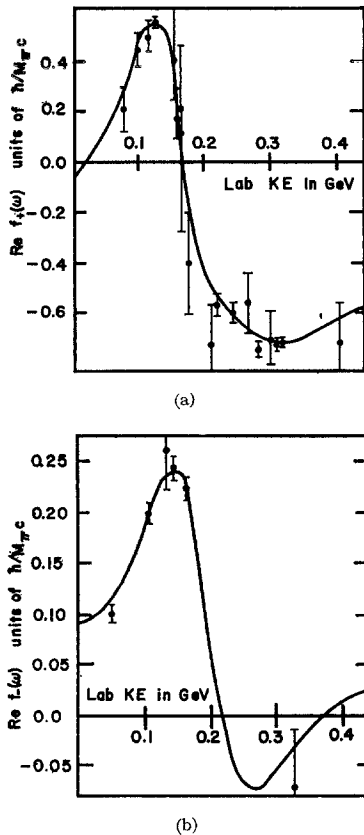


FIG. 11. The theoretical curves (solid lines) and experimental values for the real parts of the π^+-p and π^--p scattering amplitudes taken from Spearman.¹⁰

cross sections similar to those taken from Hamilton⁹ and shown in Fig. 10. Such a curve of $\text{Re } f_+(\omega)$ is shown in Fig. 11(a) along with points representing the experimental data available as of 1959. A similar curve also taken from Spearman¹⁰ is shown in Fig. 11(b) for $\text{Re } f_-(\omega)$ for which much less experimental data were then available. Within the indicated experimental errors, the agreement between theory and experiment is quite good.

In Eq. (29) the integrals are evaluated from μ to ∞ . Obviously experimental data do not exist up to infinite energies. In fact the graph of σ_T goes only up to a pion kinetic energy of about 2.0 GeV.

The effect of the portion of the integral for which data are not available was taken into account by evaluating the curves shown here assuming a constant total cross section of 30 mb for energies above 1900 MeV.

VII. CONCLUSION

Dispersion relations have had a diverse and fruitful history. First applied to the scattering of x rays in matter by Kronig¹¹ and Kramers¹² the techniques were subsequently applied to other wave lengths of electro-magnetic radiation. A recent article by Sharnoff¹³ discusses the proof of these relations for systems involving electro-magnetic radiation. In the 1930's and subsequently Bode¹⁴ and Lee¹⁵ and others applied dispersion relations to the analysis of electric networks.

Since 1950 techniques of dispersion relations have been developed for elementary particles by Goldberger, Gell-Mann, Thirring, and a growing number of other contributors.^{1,2} Some physicists feel that properly extended and generalized dispersion relations will provide the key to the analysis of elementary particles and their interactions. In a very real sense the exploitation of the usefulness of dispersion relations as a tool for physical analysis has only begun.

¹¹ R. Kronig, J. Opt. Soc. Am. **12**, 547 (1926).

¹² H. A. Kramers, Atti Congr. Intern. Fisica, Como **2**, 545 (1927).

¹³ M. Sharnoff, Am. J. Phys. **32**, 40 (1964).

¹⁴ H. W. Bode, *Network Analysis and Feedback Amplifier Design* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1945).

¹⁵ Y. W. Lee, J. Math. & Phys. **2**, 83 (1931).